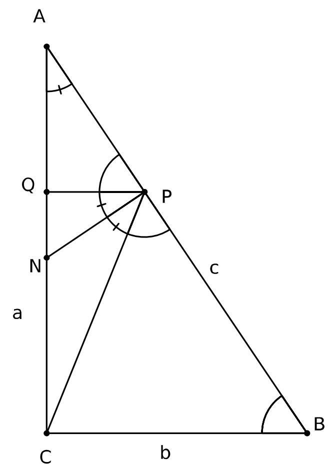

Also solved by Kenneth Korbin, NY, NY; David Stone and John Hawkins, Georgia Southern University Statesboro GA, and the proposer.

5525: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Drobeta Turnu-Severin, Mehedinti, Romania*

Find real values for x and y such that:

$$4\sin^2(x+y) = 1 + 4\cos^2 x + 4\cos^2 y.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Put $u = e^{2ix}, v = e^{2iy}$. Then the given equation reads as

$$\begin{aligned} 0 &= \left(e^{2ix+2iy} + e^{-2ix-2iy} - 2\right) + 1 + \left(e^{2ix} + e^{-2ix} + 2\right) + \left(e^{2iy} + e^{-2iy} + 2\right) = \\ &= u\frac{1}{uv} + u + \frac{1}{u} + v + \frac{1}{v} + 3 = \frac{(uv+u+1)(uv+v+1)}{uv}. \end{aligned}$$

So either $v = -\frac{1}{u} - 1$ or $\frac{1}{v} = -u - 1$. If x and y run through the real numbers v and $\frac{1}{v}$ represent circles in the complex plane with radius 1 and center 0, while $-u-1$ and $\frac{-1}{u} - 1$ represent circles with radius 1 and center -1. Therefore $(u,v) \in \left\{\left(e^{2\pi i/3}, e^{2\pi i/3}\right), \left(e^{-2\pi i/3}, e^{-2\pi i/3}\right)\right\}$ which translates to $x \equiv y \equiv \pm\frac{\pi}{3} (\text{mod } \pi)$.

Solution 2 by Michael C. Faleski, University Center, MI

Let's rewrite the statement of the problem using several trigonometric indentities. This leads to

$$4(\sin x \cos y + \sin x \cos y)^2 = 1 + 4\cos^2 x + 4\cos^2 y$$

$$4(\sin^2 x \cos^2 y + \sin^2 y \cos^2 x + 2\sin x \sin y \cos x \cos y) = 1 + 4\cos^2 x + 4\cos^2 y$$

$$4\left((1-\cos^2 x)\cos^2 y+\cos^2 x(1-\cos^2 y)+2\sin x\sin y\cos x\cos y\right)=1+4\cos^2 x+4\cos^2 y$$

$$-8\cos^2 x\cos^2 y+8\sin x\sin y\cos x\cos y=1$$

$$-8\left(\frac{1}{2}+\frac{1}{2}\cos(2x)\right)\left(\frac{1}{2}+\frac{1}{2}\cos(2y)\right)+2\sin 2x\sin 2y=1$$

$$-2(1+\cos 2x+\cos 2y+\cos 2x\cos 2y)+2\sin 2x\sin 2y=1$$

$$-2-2\cos 2x-2\cos 2y-2\cos 2x\cos 2y+2\sin 2x\sin 2y=1$$

$$-2\cos 2x-2\cos 2y-2(\cos 2x\cos 2y-\sin 2x\sin 2y)=3$$

$$\cos 2x+\cos 2y+\cos(2x+2y)=-\frac{3}{2}.$$

And now we use $\cos a=\cos b=2\cos\left(\frac{1}{2}(a+b)\right)\cos\left(\frac{1}{2}(a-b)\right)$
to produce $2\cos(x+y)\cos(x-y)+(2\cos^2(x+y)-1)=-\frac{3}{2}$,
and so we have $2\cos^2(x+y)+2\cos(x-y)\cos(x+y)+\frac{1}{2}=0$, or
$\cos^2(x+y)+\cos(x-y)\cos(x+y)+\frac{1}{4}=0$.
We will now use the quadratic formula to solve for $\cos(x+y)$.

$$\cos(x+y)=\frac{-\cos(x-y)\pm\sqrt{\cos^2(x-y)-1}}{2}.$$

As we are required to have real solutions, this means that
$\cos^2(x-y)-1\geq 0\longrightarrow\cos^2(x-y)\geq 1$. This condition is only true for
$\cos^2(x-y)=1\longrightarrow\cos(x-y)=1$.
Letting $y=x-a$, we find $\cos a=1\longrightarrow a=2n\pi,\forall n\in Z$.

$$\cos(x+y)=-\frac{\cos(x-y)}{2}=-\frac{1}{2}.$$

Since $y=\pm 2n\pi$, then for $0\leq x\leq 2\pi, x=y$. Hence, $\cos 2x=-\frac{1}{2}$, which leads to
$2x=\frac{2}{3}\pi,\frac{4}{3}\pi\longrightarrow x=\left(\frac{1}{3}\pi,\frac{2}{3}\pi\right)$. So, for $0\leq x,y\leq 2\pi,(x,y)=\left(\frac{1}{3}\pi,\frac{1}{3}\pi\right),\left(\frac{2}{3}\pi,\frac{2}{3}\pi\right)$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

$$4\sin^2((x+y)=1+4\cos^2 x+4\cos^2 y\iff 4\left(1-\cos^2(x+y)\right)=1+2\cos(2x)+2+2\cos(2y)$$

$$\iff\quad 4-4\cos^2(x+y)=5+4\cos\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2x-2y}{2}\right)$$

$$\iff\quad 0=4-4\cos^2(x+y)+4\cos(x+y)\cos(x-y)+1$$

$$\iff \quad 0 = (2\cos(x+y) + \cos(x-y))^2 - \cos^2(x-y) + 1$$

$$\iff \quad 0 = (2\cos(x+y) + \cos(x-y))^2 + \sin^2(x-y)$$

$$\iff \quad 2\cos(x+y) + \cos(x-y) = 0 = \sin(x-y) \iff x - y = k\pi,\ k \in Z$$

$$\cos(x+y) + \cos(k\pi) = 0 \iff x - y = k\pi;\ \ \cos(x+y) = \frac{(-1)^{k+1}}{2},\ k \in Z$$

$$\iff \quad x - y = k\pi;\ x + y = \arccos\frac{(-1)^{k+1}}{2},\ \in Z$$

$$\iff \quad x = \frac{1}{2}\left(\arccos\frac{(-1)^{k+1}}{2} + k\pi\right),\ y = \frac{1}{2}\left(\arccos\frac{(-1)^{k+1}}{2} - k\pi\right),\ k \in Z.$$

Solution 4 by Kee-Wai Lau, Hong Kong, China

Since $\sin(x+y) = \sin x\cos y + \cos x\sin y$, so the given equation is equivalent to $1 - 8\sin x\cos x\sin y\cos y + 8\cos^2 x\cos^2 y = 0$. Clearly $\cos x \neq 0$ and $\cos y \neq 0$. So dividing both sides of the last equation by $\cos^2 x\cos^2 y$, we obtain $\sec^2 x\sec^2 y - 8\tan x\tan y + 8 = 0$ or $(1+\tan^2 x)(1+\tan^2 y) - 8\tan x\tan y + 8 = 0$, or

$$(\tan x - \tan y)^2 + (\tan x\tan y - 3)^2 = 0.$$

Thus $\tan x = \tan y$ and $\tan x\tan y = 3$, so that $\tan x = \tan y = \sqrt{3}$ or $\tan x = \tan y = -\sqrt{3}$. It follows that

$$(x,y) = \left(\frac{\pi}{3} + m\pi, \frac{\pi}{3} + n\pi\right),\ \left(\frac{2\pi}{3} + m\pi, \frac{2\pi}{3} + n\pi\right),$$

where m and n are arbitrary integers.

Solution 5 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany

Using $\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$ we see that the equation

$$4\sin^2(x+y) = 1 + 4\cos^2(x) + 4\cos^2(y)$$

is equivalent to

$$0 = 3 + 2\cos(2x+2y) + 2\cos(2x) + 2\cos(2y) =: f(x,y).$$

Using $\sin(2a) + \sin(2b) = 2\sin(a+b)\cos(a-b)$ we obtain

$$\begin{aligned} \text{grad} f(x,y) &= -4\cdot(\sin(2x+2y) + \sin(2x), \sin(2x+2y) + \sin(2y)) \\ &= -8\cdot(\sin(2x+y)\cos y, \sin(x+2y)\cos x). \end{aligned}$$

Therefore, $\text{grad} f(x,y) = (0,0)$ happens if

- $2x = \pi \pmod{2\pi}$ and $2y = \pi \pmod{2\pi}$. The critical points $\left(\frac{2n+1}{2}\pi, \frac{2m+1}{2}\pi\right)$ with integers n, m satisfy

$$f\left(\frac{2n+1}{2}\pi, \frac{2m+1}{2}\pi\right) = 3 + 2\cdot 1 + 2(-1)^{n+1} + 2(-1)^{m+1} > 0.$$

- $2x=\pi\,(\text{mod } 2\pi)$ and $2x+y=0\,(\text{mod } \pi)$. The critical points $\left(\frac{2n+1}{2}\pi, m\pi-(2n+1)\pi\right)$ with integers n,m satisfy

$$f\left(\frac{2n+1}{2}\pi, m\pi-(2n+1)\pi\right)=3+2\cdot(-1)+2(-1)^{n+1}+2\cdot 1>0.$$

- 2y=π (mod 2π) and $x+2y=0$ (mod π) is symmetrical to the preceding case.
- $2x+y=0\,(\text{mod}\pi)$ and $x+2y=0$ (modπ). This implies $3x+3y=(n+m)\pi$ and $x-y=(n-m)\pi$ with integers n,m. We infer that $(x,y)=\frac{\pi}{3}(2n-m, 2m-n)$ are the remaining critical points of f.

$$\begin{aligned}
& f\left(\frac{2n-m}{3}\pi, \frac{2m-n}{3}\pi\right) \\
= \;& 3+2\cos\frac{2(n+m)\pi}{3}+2\cos\frac{(4n-2m)\pi}{3}+2\cos\frac{(4m-2n)\pi}{3} \\
= \;& 3+2\left(2\cos^2\frac{(n+m)\pi}{3}-1\right)+4\cos\frac{(n+m)\pi}{3}\cos(n-m)\pi \\
= \;& 1+4\cos^2\frac{N\pi}{3}+4(-1)^N\cos\frac{N\pi}{3}=\left(1+2(-1)^N\cos\frac{N\pi}{3}\right)^2\geq 0
\end{aligned}$$

with $N:=n+m$. Consequently, the function value is equal to zero iff N is not a multiple of 3.

In total, we have $f(x,y)\geq 0$ on R^2 and $f(x,y)=0$ if and only if $(x,y)=(2n-m, 2m-n)\frac{\pi}{3}$, for all integers n,m satisfying $n+m\neq 0$ (mod 3). The solutions of the above trigonometric identity are exactly the zeros of f.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; David E. Manes, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Marian Ursărescu, "Roman Vodă College," Roman, Romania, and the proposer.

5526: *Proposed by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece*

The lengths of the sides of a triangle are 12, 16 and 20. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We claim that there is exactly one straight line which simultaneously halves the area and the perimeter of the triangle.

If the line passes through the sides of length 12 and 16, and its intersection with side 12 is x units from the acute angle on that side, then the line cuts off a right triangle of base $12-x$